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Analysis of the anomalous Hall effect in a double layer of a ferromagnetic and a normal metal

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Abstract. A new method is suggested to investigate the mechanism of the anomalous Hall effect (AHE) in ferromagnetic metals. Using a double layer of a ferromagnet and a normal metal of increasing thickness one can manipulate the AHE in the ferromagnet without changing the ferromagnet's structure and electronic properties. The conduction electrons from the normal metal carry their drift velocity across the interface into the ferromagnetic film and induce an additional AHE conductance ΔG_{xy}. Its dependence on the mean free path in the normal metal distinguishes between the side jump and the skew scattering mechanisms for the AHE in the ferromagnet.

PACS. 72.10.-d Theory of electronic transport; scattering mechanisms – 72.25.Mk Spin transport through interfaces – 73.40.Jn Metal-to-metal contacts – 72.25.Ba Spin polarized transport in metals

1 Introduction

In ferromagnetic metals and metals with magnetic impurities one observes two contributions to the Hall effect, (i) the normal Hall effect and (ii) the anomalous Hall effect (AHE). The AHE is caused by spin-orbit scattering through the interaction of the conduction-electron spin with the magnetic moments of the sample. The anomalous Hall effect was already observed by Hall more than a century ago [1]. However, theoretically it is a rather complicated problem. There are two main mechanisms discussed in the literature, (a) skew scattering and (b) side jump. Both require a scattering mechanism for the conduction electrons and vanish in pure samples. The first models of skew scattering were developed by Karpulus and Luttinger [2] and Smit [3], while the side jump was proposed by Berger [4]. Due to its possible application in spintronics, the anomalous Hall effect has experienced a renewed interest during the past years [5–8] (for further references see [9]). Recently an additional mechanism has been under discussion which is connected with the Berry phase and believed to occur even in the absence of any scattering (see for example [10]). Here we restrict the discussion to the skew scattering and the side jump.

The theory of the AHE is far from mature. For example, some years ago our group measured the AHE of vanadium impurities in alkali films [11]. The AHE was positive in Cs and Rb hosts, close to zero in K hosts and negative in Na hosts. Nobody understands the sign of the AHE in these systems so far. The theory of a physical phenomenon cannot be considered satisfactory if one cannot even predict the sign of the effect. Therefore an analysis of the AHE is quite desirable. In particular a clear understanding of the importance of the different mechanism is needed.

Skew scattering is due to the anisotropic scattering of the conduction electrons by magnetic scattering centers. The amplitude of the scattered wave depends on the scattering angles (θ, ϕ) and shows left-right asymmetry. For example if an electron with momentum $\mathbf{k} = (k_x, 0, 0)$ in the x-direction is scattered by a magnetic moment (polarized in the z-direction) then the integrated momentum of the scattered wave has a finite component in the $(-y)$-direction.

For the side jump the electron does not propagate in the $(-y)$-direction after the scattering but the whole scattered electron is displaced by the distance Δy in the $(-y)$-direction. Both mechanisms yield an AHE.

It is often stated that for skew scattering the anomalous Hall resistivity ρ_{yx} is proportional to the resistivity ρ_{xx} while for the side jump ρ_{yx} is proportional to the square of the resistivity ρ_{xx}^2. A number of experimental investigations applied this power law $\rho_{yx} \propto \rho_{xx}^p$ with $p = 1$ for skew scattering and $p = 2$ for the side jump to identify the mechanism of the AHE by changing the resistivity of their samples and analyzing the dependence of ρ_{yx} on ρ_{xx}.

On the other hand this power law between ρ_{xx} and ρ_{yx} is only derived for a single spin and might not be fulfilled in a real ferromagnet. Furthermore changing the resistivity of a ferromagnet can alter the magnetic scattering in complex ways. It would be of considerable value to find an experiment that distinguishes between skew scattering

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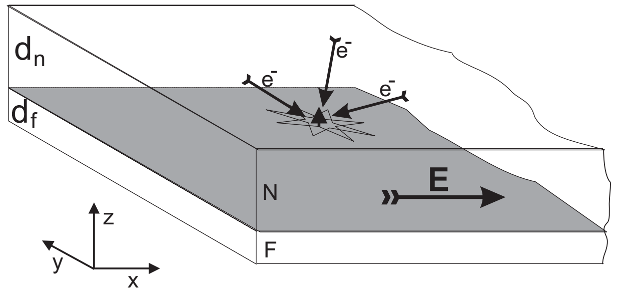

Fig. 1. A double layer consisting of a ferromagnetic film F and a normal metal film N. In the presence of an electric field the conduction electrons in N carry their larger drift velocity into the lower layer F and create a large anomalous Hall effect (AHE) in F. Its dependence on the mean free path in N identifies the origin of the AHE.

and side jump without changing the structure of the ferromagnet.

In this paper I suggest a new approach to investigate the AHE of a ferromagnetic film experimentally. A thin ferromagnetic film is used as the target of a scattering experiment by exposing it to incident electrons. The momentum of the incident electrons is varied and the electrons are scattered by the magnetic moments. The (integrated) angular scattering intensity is measured. This appears to be a conventional scattering experiment but it is really a measurement of the transport properties in a double layer consisting of a ferromagnet and a normal metal. The probing electrons are the conduction electrons of the normal metal film which cross the interface into the ferromagnetic film.

Figure 1 shows the geometry and the idea of the proposed experiment. A double layer consisting of a ferromagnetic film F and a normal metal film N is prepared. The normal metal film is condensed on top of the ferromagnetic film. Its thickness and its mean free path (MFP) can be increased in situ. The ferromagnetic film is prepared with a much shorter MFP than the normal metal. This simplifies the underlying physics and the evaluation of the experiment.

In the presence of an electric field E (in the x-direction) the electrons accumulate finite drift velocities in the normal metal and the disordered ferromagnet. The electrons of both metals cross the interface. The electrons which cross from the normal metal into the ferromagnet increase the current density in the upper layers of ferromagnetic film dramatically because they carry a much larger drift velocity. This injected high current density in the ferromagnet is proportional to the MFP in the normal metal. It creates an additional AHE in F. If the AHE is due to the side-jump mechanism then the injected current yields an AHE conductance which is proportional to the MFP l_n in the normal metal. If the AHE is due to skew scattering then a large fraction of the scattered electrons returns into the normal metal and propagates there the distance l_n. This contribution to the AHE conductance is proportional to the square of the MFP in the normal metal. By changing the MFP in the normal metal one can analyze the origin of the AHE in the ferromagnet without changing the structure of the ferromagnet.

2 Skew scattering and side jump in a ferromagnet

Since we have to calculated the AHE of an F/N double layer (F stands for ferromagnet and N for normal metal) we briefly recall the AHE resistivity for skew scattering and the side jump in a ferromagnetic sample. The magnetic moments may be aligned in the z-direction. The sample is disordered and has a finite ρ_{xx}.

We treat the ferromagnet in the following model: the conduction process is essentially carried by the spin-up and -down s-electrons. I denote their densities as n_σ, their Fermi velocities as v_σ and their relaxation times as τ_σ. Furthermore the s-electrons are treated as quasi-free electrons with an effective mass m_σ, so that $v_\sigma = \hbar k_\sigma / m_\sigma$. Since the d-electrons with their flat bands barely participate in the conduction process their contribution to the current is neglected. The d-bands with their large density of states serve mainly as final states for the scattered s-electrons. Since the d-density of states for spin-up and -down electrons is different one obtains different scattering times $\tau_\uparrow, \tau_\downarrow$ for spin-up and -down electrons. For our purpose it is even more important that the d-scattering in connection with the spin-orbit interaction in the d-states is the essential source of the AHE. In this paper we are interested in the effect of a specific sample arrangement on the AHE. The AHE itself is described by phenomenological parameters and not calculated from first principles

In a disordered ferromagnet the scattering will be partially potential scattering, i.e. spin-independent, and partially magnetic or spin-dependent scattering. We describe the potential scattering centers by their concentration n_i and their total scattering cross section a_i (the index i for impurity). Similarly the magnetic scattering centers have the concentration n_m with the scattering cross section $a_{m\sigma}$ (which depends on the spin σ of the conduction electrons). Then the mean free paths l_σ and the relaxation times τ_σ of the conduction electrons with spin σ are

$$\frac{1}{l_\sigma} = (n_i a_i + n_m a_{m\sigma}), \frac{1}{\tau_\sigma} = \frac{v_\sigma}{l_\sigma}$$

yielding the longitudinal resistivity

$$\rho_{xx,\sigma} = \frac{m_\sigma}{n_\sigma e^2 \tau_\sigma}.$$

Next we consider the transverse resistivity ρ_{xy}. We begin with the skew scattering mechanism. In Figure 2a an electron with spin σ propagates in the x-direction. The wave experiences skew-scattering by the d-states. The integrated momentum of the skew-scattered wave possesses an electron momentum in the negative y-direction with the weight $a_{AH,\sigma}$. Here $a_{AH,\sigma}$ is the AHE cross section. It is defined so that $-\hbar k_F a_{AH,\sigma}$ is equal to the $(-y)$-component of the integrated momentum of the

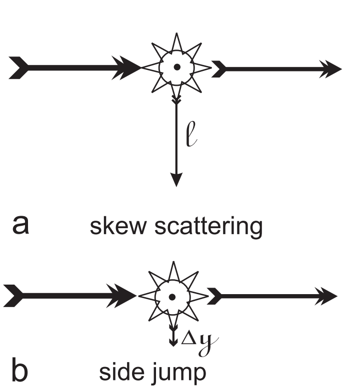

Fig. 2. An (spin-up) electron wave with momentum $\mathbf{k} = (k_x, 0, 0)$ propagates in the x-direction. A part $a_{m\sigma}$ of the wave is skew-scattered by a magnetic moment and carries a momentum in the $(-y)$-direction. (b) This time the side jump displaces the scattered electrons wave by Δy_σ in the $(-y)$-direction.

scattered wave. (A possible forward scattering should be incorporated into the scattering cross section $a_{m\sigma}$.) Any current in the x-direction generates a current in the $(-y)$-direction with a transition rate of $dj_{y,\sigma}/dt = -j_{x,\sigma}/\tau_{AH,\sigma}$ where $1/\tau_{AH,\sigma} = n_{m,\sigma} a_{AH,\sigma} v_\sigma$.

One obtains the following rate equations for the current in the (x, y)-plane

$$\frac{dj_{x,\sigma}}{dt} = -j_{x,\sigma} v_\sigma \left(n_i a_i + n_m a_{m\sigma}\right) + j_{y,\sigma} v_\sigma n_m a_{AH,\sigma} + \frac{e^2 n_\sigma}{m_\sigma} E_x$$

$$\frac{dj_{y,\sigma}}{dt} = -j_{x,\sigma} v_\sigma n_m a_{AH,\sigma} - j_{y,\sigma} v_\sigma \left(n_i a_i + n_m a_{m\sigma}\right) + \frac{e^2 n_\sigma}{m_\sigma} E_y.$$

In the steady state one has $d\mathbf{j}_\sigma/dt = 0$ and obtains the resistivity tensor

$$(\rho) = \frac{m_\sigma}{e^2 n_\sigma} \frac{1}{\tau_\sigma} \begin{pmatrix} 1 & -l_\sigma n_m a_{AH,\sigma} \\ l_\sigma n_m a_{AH,\sigma} & 1 \end{pmatrix}.$$

The transverse resistivity $\rho_{yx,\sigma}$ for the spin σ is

$$\rho_{yx,\sigma} = \rho_{xx,\sigma} l_\sigma n_m a_{AH,\sigma} = \rho_{xx,\sigma} \frac{n_m a_{AH,\sigma}}{n_i a_i + n_m a_{m\sigma}}. \qquad (1)$$

In Appendix B equation (1) is derived with a simple hand-waving argument.

For the discussion below of the AHE in a F/N double layer we keep the following result in mind: when the electric field in the x-direction generates a current density $j_{x,\sigma}$ then the skew scattering provides a current source in the $(-y)$-direction with the strength $dj_{y,\sigma}/dt = -j_{x,\sigma} v_\sigma n_m a_{AH,\sigma}$.

Equation (1) yields the well known statement that the AHE resistivity $\rho_{yx,\sigma}$ due to skew scattering is proportional to the resistivity $\rho_{xx,\sigma}$, for a single spin direction! Since the electron has two spins we have to (i) invert the resistivity tensors $(\rho_{ij})_{\uparrow,\downarrow}$ for each spin to obtain the conductivity tensors $(\sigma_{ij})_{\uparrow,\downarrow}$, (ii) add the two conductivity tensors and (iii) invert the resulting tensor.

$$(\rho) = \left((\rho)_\uparrow^{-1} + (\rho)_\downarrow^{-1}\right)^{-1}. \qquad (2)$$

Since the longitudinal resistivities $\rho_{xx,\sigma}$ of spin-up and -down electrons in the ferromagnet are generally quite different the original linearity between $\rho_{yx,\sigma}$ and $\rho_{xx,\sigma}$ for the individual spin might be replaced by a more complicated dependence.

For the side jump the electron does not propagate in the $(-y)$-direction after the scattering but the whole scattered electron is displaced by the distance Δy_σ in the $(-y)$-direction. One can compare this displacement of the electron in the side jump with the effect of the skew scattering. In a skew scattering event a fraction $a_{AH,\sigma}$ of an electron is "displaced" by the MFP l_σ in the $(-y)$-direction. Therefore one obtains the off-diagonal AHE resistivity of the side jump by replacing $a_{AH,\sigma} l_\sigma$ in equation (1) by $a_{m\sigma} \Delta y_\sigma$. This yields for the side jump

$$\rho_{yx,\sigma} = \rho_{xx,\sigma} \Delta y_\sigma n_m a_{m,\sigma}. \qquad (3)$$

If $\rho_{xx,\sigma}$ is proportional to the density of magnetic scattering centers n_m then $\rho_{yx,\sigma}$ is proportional to the square of $\rho_{xx,\sigma}$ if the scattering cross section $a_{m\sigma}$ and the side jump are independent of the resistivity, for a single spin direction! Again, using equation (2) for the total AHE resistivity might interfere with the quadratic relation. Furthermore the parameters a_i, $a_{m\sigma}$, $a_{AH,\sigma}$ Δy_σ are not independent of the disorder. The scattering potential is generally not the atomic potential but the deviation from the periodic potential. This potential is generally not spherically symmetric but is rather the gradient of a spherical potential. All of the scattering parameters a_i, $a_{m\sigma}$, $a_{AH,\sigma}$ and Δy_σ may change in complicated ways with the disorder or alloying of the ferromagnet. In particular the behavior of Δy_σ as a function of disorder is very critical. As Berger showed the side jump, which is caused by the spin-orbit interaction, only becomes significant because the spin-orbit interaction in the magnetic atoms can be enhanced by a large factor which he estimated to be of the order of 3×10^4. Even the smallest change in the local environment of the magnetic atom could change this enhancement factor.

For the discussion below of the AHE in a F/N double layer we keep the following result in mind: when the electric field in the x-direction generates a current density $j_{x,\sigma}$ then the side jump slightly shifts the direction of the current into the $(-y)$-direction. The resulting current component in the $(-y)$-direction, $-\Delta y_\sigma n_m a_{m,\sigma}\, j_{x,\sigma}$, is local. In contrast skew scattering generates a non-local $(-y)$-component of the current which propagates away from the scattering center.

We summarize: the simple power law dependence of ρ_{yx} on the resistivity ρ_{xx} for skew scattering and side jump may not be reliable for the following reasons:

- the contribution of two kinds of electrical carriers in ferromagnets, spin up and down electrons, destroys the simple relation between ρ_{yx} and ρ_{xx};
- the scattering parameters a_i, $a_{m\sigma}$, $a_{AH,\sigma}$ and Δy_σ will change in complicated ways with the disorder or alloying of the ferromagnet;
- the large enhancement of the spin-orbit interaction (which determines the side-jump parameter) might be sensitive to the disorder.

3 Skew scattering and side jump in a double layer of ferromagnet and normal metal

Now we return to the geometry in Figure 1 and calculate the AHE conductance of a F/N (ferromagnet/nonmagnetic metal) double layer. As before we assume that the conduction in the ferromagnet is carried by the s-electrons. The thickness and MFPs of the ferromagnetic film are denoted as d_f, l_σ for spin σ and d_n, l_n for the normal metal film. Because the ferromagnetic and normal metal films are in parallel their conductances would simply add if there would be no interface crossing between the films. Without the crossing the normal metal film would not contribute to the AHE.

When electrons cross the interface from the ferromagnet into the normal metal the drift velocity of spin-up and -down electrons can be different. Then the electrons carry a spin current into the normal metal. If the normal metal possesses a strong spin-orbit scattering then the spin current itself can cause an AHE in the normal metal [12]. In the present paper this would complicate the experiment. Therefore we choose a normal metal with sufficient small spin-orbit scattering (for example the alkali metals fulfill this requirement perfectly).

The next task is to determine the interface crossing. The density of s-electrons in $3d$-ferromagnets is surprisingly small. For example for Fe the Fermi energies of s-electrons with majority and minority spins are given in the literature as 2.25 eV and 0.5 eV [13]. We consider here the case when the Fermi energies in the ferromagnet are smaller than in the normal metal. Quasi-classically all (s-)electrons in the ferromagnet which move towards the interface ($k_z > 0$) would cross into the normal metal without reflection. In wave mechanics one obtains a small reflection at the interface which can be taken care of by a transmission factor $t \leq 1$. During the calculation we set $t = 1$; the effect of a smaller transmission is discussed below.

Since all electrons in F close to the interface with $k_z > 0$ leave the ferromagnet the same number of electrons has to return from N. That means that all electrons with $k_z < 0$ in the ferromagnet close to the interface have crossed the interface from the normal metal. In the presence of an electric field in the x-direction these carry the large drift velocity from the normal metal and inject a large current density in the top layer of the ferromagnet. We calculate this injected current by applying the (linearized) Boltzmann equation using Chamber's method of the vector mean free path (VMFP) [14]. The method of the VMFP is sketched in the appendix together with the calculation of the injected currents for spin-up and -down in the top layers of the ferromagnet.

The injected current (per width of the film) for spin σ is

$$I_x, \sigma = \frac{1}{16}\frac{e^2\tau_n}{m_n}\left(N_\sigma m_\sigma v_{F,\sigma}^3\tau_\sigma\right)E = \frac{e^2 n_\sigma \tau_n}{m_n}\frac{3}{16} l_\sigma E$$

where N_σ is the density of electron states per spin in the ferromagnet. This current flows in a thin layer of F whose thickness is roughly half the MFP, i.e., $l_\sigma/2$. It is proportional to the relaxation time in the normal metal. The interface conductance $G^{\mathrm{i}}_{xx,\sigma}$ due to the injected electrons with spin σ is the product of the conductivity $e^2 n_\sigma \tau_n / m_n$ times the thickness of $(3/16)\, l_\sigma$,

$$G^{\mathrm{i}}_{xx,\sigma} = \frac{e^2 n_\sigma \tau_n}{m_n}\frac{3}{16} l_\sigma.$$

The conductivity consists of the electron density n_σ of the ferromagnetic spin component and the relaxation time and effective mass of the normal conductor. The resulting longitudinal part of the conductance $G_{xx} = I_{in}/E$ is similar to the results by Fuchs [15] and Sondheimer [16] for thin films but extended to sandwiches.

The injected current yields an additional large AHE. The resulting contribution to the anomalous Hall conductance depends on the mechanism of the AHE.

Side jump: the electrons which carry the injected current $I^{\mathrm{i}}_{x,\sigma}$ in the ferromagnet contribute to the side jump. The direction of the current is locally shifted yielding a current component of $\Delta y_\sigma n_m a_{m,\sigma} I^{\mathrm{i}}_{x,\sigma}$ in the $(-y)$-direction. Both spins together produce an additional interface AHE conductance

$$G^{\mathrm{i}}_{xy} = \frac{1}{16}\frac{e^2\tau_n}{m_n}\sum\nolimits_\sigma N_\sigma m_\sigma v_{F,\sigma}^3 \tau_\sigma \Delta y_\sigma n_m a_{m,\sigma}.$$

For the side jump the AHE conductance at the interface is proportional to the MFP l_n of the electrons in the normal metal film. The electrons which cross from the ferromagnet to the normal metal do not contribute to the AHE (for the side jump mechanism).

Skew scattering: in contrast to the side jump, here part of the important physics happens after the scattering because half of the skew-scattered electrons propagate back towards the normal film. Through the skew scattering a current in the $(-y)$-direction is created with the rate

$$\frac{dI_{y,\sigma}}{dt} = -v_\sigma n_m a_{AH,\sigma} I_{x,\sigma}.$$

In the ferromagnet this current decays with the rate $1/\tau_\sigma$. A fraction $\beta/2$ of the skew-scattered electrons propagate back into the normal metal. This factor is due to the fact that only half of the scattered electrons move back towards the normal metal. Since they are roughly the distance $l_{\uparrow}/2, l_{\downarrow}/2$ from the interface only a fraction β reaches the normal metal without being scattered in the ferromagnet. The factor β is less than one and of the order of 1/2. (If the scattering in the ferromagnet would be isotropic then β would have the value 1/2.) In the normal metal the current $I^{\mathrm{i}}_{y,\sigma}$ decays with the much smaller rate $1/\tau_n$ and takes the stationary value

$$I^{\mathrm{i}}_{y,\sigma} = -\frac{\beta}{32}\frac{e^2\tau_n^2}{m_n} n_m \left(a_{AH,\sigma} N_\sigma m_\sigma v^4_{F,\sigma}\tau_\sigma\right) E = G^{\mathrm{i}}_{xy,\sigma} E.$$

This yields a contribution to the AHE conductance which is proportional to τ_n^2, i.e. to the square of the MFP in the normal metal,

$$G^{\mathrm{i}}_{xy} = \frac{\beta}{32}\frac{e^2\tau_n^2}{m_n} n_m \sum\nolimits_\sigma \left(a_{AH,\sigma} N_\sigma m_\sigma v^4_{F,\sigma}\tau_\sigma\right).$$

(The conduction electrons which are accelerated in the ferromagnet and cross into the normal metal after the scattering also yield a contribution to the AHE. However, this contribution is smaller than $G^{\mathrm{i}}_{xy,\sigma}$ by the ratio τ_σ/τ_n.)

For skew scattering the additional anomalous Hall conductance G^{i}_{xy} at the interface is proportional to the square of the MFP l_n^2 in the normal metal.

4 Conclusion

In this paper the author suggests the investigation of the AHE in double layers of a ferromagnetic and a normal metal film. The MFP in the normal metal N should be much larger than in the ferromagnet F. An electric field in the x-direction causes different drift velocities of the conduction electrons in the N and F. Electrons which cross from N to F carry a much larger drift velocity. These electrons introduce a large current density parallel to the electric field at the top layers of F (within the thickness of about half the MFP of spin-up and -down electrons, $l_{\uparrow}/2$ and $l_{\downarrow}/2$). The resulting current is calculated using the vector mean free path method. It is proportional to the MFP in the normal metal. The MFP in N can be modified by increasing the film thickness of N in situ during a single experiment. The properties of the ferromagnetic film remain unchanged.

The injected current experiences scattering by the magnetic scattering centers in F. This generates an AHE current component in the $(-y)$-direction. It yields an additional contribution G^{i}_{xy} to the AHE conductance. The two proposed mechanisms for the AHE, side jump and skew scattering, yield different dependences of G^{i}_{xy} on the MFP l_n in the normal metal:

- Side jump: $G^{\mathrm{i}}_{xy} = \mathrm{const}_1 \times l_n$.
- Skew scattering: $G^{\mathrm{i}}_{xy} = \mathrm{const}_2 \times l_n^2$.

The value of the constants is calculated for perfect transmission through the interface from F to N. A reduced transmission will reduce the constants but does not change the dependences on the MFP l_n in the normal metal.

In the performance of such an experiment one has to vary the mean free path of the conduction electrons in the normal metal. This can be, for example, achieved by successively increasing the thickness of the normal metal. The MFP generally increases with the film thickness. Then one has to plot the total anomalous Hall conductance G_{xy} of the F/N double layer versus l_n and l_n^2. For the side jump the AHE conductance G^{i}_{xy} at the interface is proportional to l_n while for skew scattering it is proportional to l_n^2.

This method has the great advantage that it leaves the structure, scattering properties and all other parameters of the ferromagnetic film unchanged. Only the MFP in the normal metal is altered (generally increased through successive evaporations). This has a considerable advantage over the traditional method where the MFP of the ferromagnet is changed by alloying or other methods.

Abbreviations: AHE = anomalous Hall effect, MFP = mean free path, VMFP = vector mean free path.

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Appendix A: Vector mean free path and relaxation time method

In the presence of an electric field $\mathbf{E} = (E, 0, 0)$ the Fermi surface is shifted and the Fermi distribution function $f_{\mathbf{k}}$ differs from the equilibrium distribution $f^0_{\mathbf{k}}$ by $g_{\mathbf{k}} = f_{\mathbf{k}} - f^0_{\mathbf{k}}$ (see Fig. C.1). The resulting current density is

$$\mathbf{j} = (-e) \sum\nolimits_\sigma \int \frac{d^3\mathbf{k}}{(2\pi)^3} g_{\mathbf{k}} \mathbf{v}(\mathbf{k}).$$

Chambers expressed $g_{\mathbf{k}}$ in terms of the vector mean free path (VMFP) $\mathbf{L}(\mathbf{k})$ [14],

$$g_{\mathbf{k}} = (-e)\, \mathbf{E}\cdot\mathbf{L}(\mathbf{k}) \left(-\frac{\partial f^0_{\mathbf{k}}}{\partial E}\right).$$

The physical meaning of $\mathbf{L}(\mathbf{k})$ can be understood as follows. We consider an assembly of electrons at the time t at $\mathbf{r}$ in volume element $d^3\mathbf{r}$ with wave vector $\mathbf{k}$ in the $\mathbf{k}$-increment $d^3\mathbf{k}$. We follow the semi-classical path $\mathbf{r}(t')$ of these electrons back in time for $t' < t$. The velocity and wave vector along this path are $\mathbf{v}(\mathbf{k}, t')$ and $\mathbf{k}(t')$ as a functions of time. Along the path electrons are continuously scattered out of and into the state $\mathbf{k}(t')$. At the time $t' < t$ in the time interval $(t', t' + dt')$ there is the probability that the fraction $dt'/\tau(t')$ electrons are scattered into the state $\mathbf{k}(t')$. The chance that these electrons propagate along the path $\mathbf{r}(\bar{t})$ $(t' < \bar{t} < t)$ without being scattered until the time t when they reach the point $\mathbf{r}$ is $P(t, t')$ (see also Ashcroft and Mermin [17])

$$P(t, t') = \exp\left(-\int_{t'}^{t} \frac{d\bar{t}}{\tau(\bar{t})}\right)$$

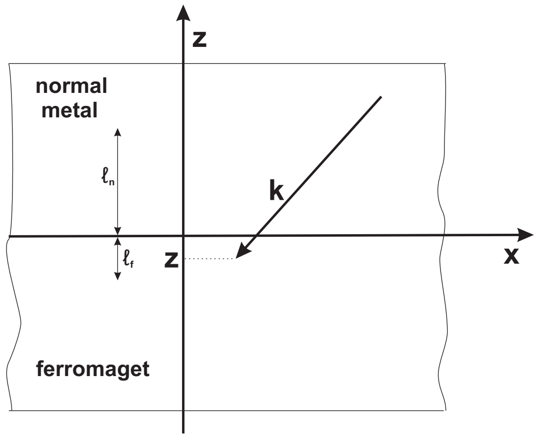

Fig. A.1. The propagation of an electron from the normal metal into the top of the disordered ferromagnet.

and $\tau\left(\bar{t}\right)$ is the relaxation time at the position $\mathbf{r}\left(\bar{t}\right)$ and in the state $\mathbf{k}\left(\bar{t}\right)$.

The VMFP is then defined as

$$\mathbf{L}\left(\mathbf{k}\right)=\int_{-\infty}^{t}P\left(t,t'\right)\mathbf{v}\left(\mathbf{k},t'\right)dt'.$$

The graphical meaning of the VMFP $\mathbf{L}\left(\mathbf{k}\right)$ is very simple. If we reverse time and start an electron from position $\mathbf{r}\left(t\right)$ with wave vector $\left(-\mathbf{k}\right)$ and velocity $-\mathbf{v}\left(\mathbf{k}\right)$ then the average distance this electron travels is equal to the negative VMFP $-\mathbf{L}\left(\mathbf{k}\right)$ of this electron. (In the presence of a magnetic field the latter has to be reversed as well.)

If we consider an electron in the normal film with $k_z<0$ (moving downward) at the interface with the ferromagnet, its VMFP $\mathbf{L}_n\left(\mathbf{k}\right)$ is given by

$$\begin{aligned}\mathbf{L}_n\left(\mathbf{k},z=0^+\right)&=\int_{t_0}^{0}\mathbf{v}\left(\mathbf{k}\right)e^{-\left|t'\right|/\tau_n}dt'\\&=\mathbf{v}\left(\mathbf{k}\right)\tau_n\left(1-e^{-d_n/v_z\tau_n}\right)\end{aligned}$$

where $t_0=-d_n/v_z=-d_nm/\left(\hbar k_z\right)$ is the time for the electron $\mathbf{k}$ to travel from the upper surface of N to the interface, $\mathbf{v}\left(\mathbf{k},t\right)=\mathbf{v}\left(\mathbf{k}\right)$ is independent of time and diffuse scattering at the upper surface is assumed. If the thickness d_n of the normal metal is larger than the mean free path l_n in N then the absolute length of $\mathbf{L}_n\left(\mathbf{k}\right)$ is $\left|\mathbf{L}_n\left(\mathbf{k}\right)\right|=l_n$. We treat this case in our calculation and consider below in the discussion the finite size effect of $d_n<l_n$.

In the next step of the calculation we consider the electrons in the very top of the ferromagnetic film with $k_z<0$ (moving away from the interface). We start with the assumption that all these electrons passed through the interface from the normal metal. During the crossing the wave vector changes from $\mathbf{k}_n$ to $\mathbf{k}_\sigma$. But the (x,y)-component of the wave vector is conserved and only the k_z component is changed so that energy is conserved. This means the $\mathbf{E}\cdot\mathbf{k}_n=\mathbf{E}\cdot\mathbf{k}_\sigma$. Since we use a ferromagnet with much shorter relaxation times τ_σ than τ_n the drift velocity of the injected electrons will quickly decay as a function of $z<0$ in the ferromagnet. We obtain for $\mathbf{L}\left(\mathbf{k},z\right)$ with $k_z<0, z<0$:

$$\mathbf{L}_\sigma\left(\mathbf{k},z\right)=\frac{\hbar\mathbf{k}_n}{m_n}\tau_n e^{-|z|/v_z\tau_\sigma}+\frac{\hbar\mathbf{k}_\sigma}{m_\sigma}\tau_\sigma\left(1-e^{-|z|/v_z\tau_\sigma}\right). \tag{4}$$

Here we assumed a linear relation between velocity and wave vector with effective masses, m_n and m_σ.

The current density in the ferromagnet is in terms of the VMFP

$$\mathbf{j}_\sigma\left(z\right)=e^2\int\frac{d^3\mathbf{k}}{\left(2\pi\right)^3}\left(\mathbf{E}\cdot\mathbf{L}_\sigma\left(\mathbf{k},z\right)\right)\left(-\frac{\partial f_{\mathbf{k}}^0}{\partial E}\right)\frac{\hbar\mathbf{k}_\sigma}{m_\sigma}. \tag{5}$$

The second term in equation (4) when introduced into equation (5) yields just the current density of the isolated ferromagnetic film with diffuse surface scattering. The first term, however, yields the current which is injected from the normal metal film into the ferromagnetic film. Its contribution to the current density is

$$\begin{aligned}\mathbf{j}_\sigma\left(z\right)=e^2\int_{k_z<0}&\frac{d^3\mathbf{k}_\sigma}{\left(2\pi\right)^3}\left(\mathbf{E}\cdot\frac{\hbar\mathbf{k}_n}{m_n}\right)\tau_n\\&\times e^{-|z|/v_z\tau_f}\left(-\frac{\partial f_{\mathbf{k}}^0}{\partial E}\right)\frac{\hbar\mathbf{k}_\sigma}{m_\sigma}.\end{aligned}$$

Using the relation $\mathbf{E}\cdot\mathbf{k}_n=\mathbf{E}\cdot\mathbf{k}_\sigma$ we obtain

$$\begin{aligned}\mathbf{j}_\sigma\left(z\right)=\frac{e^2\hbar^2\tau_n}{m_nm_\sigma}\int_{k_z<0}&\frac{d^3\mathbf{k}_\sigma}{\left(2\pi\right)^3}\left(\mathbf{E}\cdot\mathbf{k}_\sigma\right)\\&\times e^{-|z|/v_z\tau_f}\left(-\frac{\partial f_{\mathbf{k}}^0}{\partial E}\right)\mathbf{k}_\sigma.\end{aligned}$$

Only the x-component contributes to the current,

$$\begin{aligned}j_{x,\sigma}\left(z\right)=&\frac{e^2\hbar^2\tau_nk_{F,\sigma}^2}{m_nm_\sigma}N_\sigma\frac{1}{4\pi}\int_{\pi/2}^{\pi}\sin\theta d\theta\\&\times\int_0^{2\pi}d\phi\sin^2\theta\cos^2\phi\exp\left(-\frac{m_\sigma\left|z\right|}{\hbar k_{F,\sigma}\tau_\sigma\cos\left(\theta\right)}\right)E\end{aligned}$$

where N_σ is the density of states s-electrons with spin σ in the ferromagnet.

Integration over $dz,d\phi$ and $d\theta$ yields the total injected current per width W of the film (per spin),

$$\begin{aligned}I_{x,\sigma}&=\frac{e^2\hbar^2\tau_nk_{F,\sigma}^2}{m_nm_\sigma}\frac{\hbar k_{F,\sigma}\tau_\sigma}{m_\sigma}N_\sigma\frac{1}{4\pi}\\&\quad\times\int_{\pi/2}^{\pi}\sin\theta\sin^2\theta\cos\left(\theta\right)d\theta\int_0^{2\pi}\cos^2\phi d\phi E\\&=\frac{e^2\hbar^2\tau_nk_{F,\sigma}^2}{m_nm_\sigma}\frac{\hbar k_{F,\sigma}\tau_\sigma}{m_\sigma}N_\sigma\frac{1}{16}E\\&=\frac{1}{16}e^2N_\sigma v_{F,\sigma}^3\tau_n\tau_\sigma\frac{m_\sigma}{m_n}E=\frac{e^2n_\sigma\tau_n}{m_n}\frac{3l_\sigma}{16}E.\end{aligned}$$

If we take a square film with length $L=W$ then the applied voltage is EL and the resulting conductance is

$$G_{xx}=\frac{1}{16}e^2N_\sigma v_{F,\sigma}^3\tau_n\tau_\sigma\frac{m_\sigma}{m_n}=\frac{e^2n_\sigma\tau_n}{m_n}\frac{3l_\sigma}{16}.$$

Appendix B: Hand-waving derivation of AHE resistivities

Let us consider an electron moving in the x-direction in the ferromagnetic film. We draw two planes perpendicular to the direction of the electron momentum, one at $x = 0$ and the other at $x = l_\sigma$. The planes cover an area A. When we project the scattering cross sections of all scattering centers between the two planes in the volume Al_σ onto the plane at $x = l_\sigma$ the potential scattering centers yield an area $n_i A l_\sigma a_i$ and the magnetic scattering centers yield an area $n_m A l_\sigma a_{m,\sigma}$. Together the total area is $Al_\sigma (n_i a_i + n_m a_{m,\sigma})$. Inserting l_σ the combined area is equal to A, which expresses the fact that after propagating the distance of the MFP the chance to hit an impurity cross section is equal to one. The total projected area of the AHE cross section covers the fraction $l_\sigma n_m a_{AH,\sigma}$ of the area A. Therefore the total probability for the electron to transfer its momentum into the $(-y)$-direction during its lifetime τ_σ is $l_\sigma n_m a_{AH,\sigma}$. The ratio of the two currents is

$$\frac{j_{-y,\sigma}}{j_{x,\sigma}} = \frac{\sigma_{xy,\sigma}}{\sigma_{xx,\sigma}} = \frac{\rho_{yx,\sigma}}{\rho_{xx,\sigma}} = l_\sigma n_m a_{AH,\sigma}.$$

Appendix C: Averaged cross sections

In the paper we switch between the discussion of a current in the x-direction and a momentum $\mathbf{k} = (k, 0, 0)$ in the x-direction. As Figure C.1 points out the current in the x-direction is carried by the $\mathbf{k}$-states in the surface of the Fermi sphere whose occupation is altered by the shift of the Fermi surface (the area in Fig. C.1 which is marked with the arrows). All these states have Fermi momenta. The ones on the right side yield a positive contribution and those on the left a negative contribution. These different states have different cross sections.

If $f(\mathbf{k}, \mathbf{k}') = f(\theta_\mathbf{k}, \phi_\mathbf{k}; \theta_{\mathbf{k}'}, \phi_{\mathbf{k}'})$ is the scattering amplitude of a plane wave $e^{i\mathbf{kr}}$ in the direction $\mathbf{k}'$ then the AHE cross section for an electron state in the x-direction $(k_F, 0, 0)$ is

$$a_{AH,\sigma} = \int \frac{d\Omega_{\mathbf{k}'}}{4\pi} \left| f\left(\frac{\pi}{2}, 0; \theta_{\mathbf{k}'}, \phi_{\mathbf{k}'}\right)\right|^2 \sin\theta_{\mathbf{k}'} \sin\phi_{\mathbf{k}'}.$$

For the current in the x-direction which carries the same momentum as the state $(k_F, 0, 0)$ the contribution of the states $\mathbf{k} = (k, \theta_\mathbf{k}, \phi_\mathbf{k})$ to the current is equal to $(3/4\pi)\, k_F \sin\theta_\mathbf{k} \cos\phi_\mathbf{k} d\Omega_\mathbf{k}$. Here the averaged AHE cross section is

$$\overline{a_{AH,\sigma}} = 3 \int \frac{d\Omega_\mathbf{k}}{4\pi} \frac{d\Omega_{\mathbf{k}'}}{4\pi} \sin\theta_\mathbf{k} \cos\phi_\mathbf{k} \left| f(\theta_\mathbf{k}, \phi_\mathbf{k}; \theta_{\mathbf{k}'}, \phi_{\mathbf{k}'})\right|^2 \sin\theta_{\mathbf{k}'} \sin\phi_{\mathbf{k}'}.$$

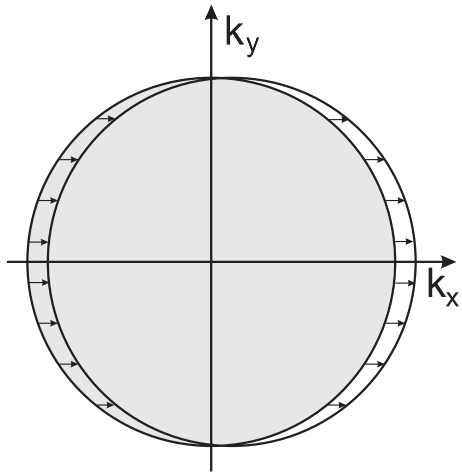

Fig. C.1. Shifted Fermi surface carrying a current in the x-direction.

In all final results the averaged cross sections have to be used. The scattering of the $\mathbf{k}$-state in the x-direction is only introduced to simplify the discussion of the underlying physics (avoiding the complication of including all the states which contribute to the current).

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